# PROPAGATION OF STRESS WAVE IN A FUNCTIONALLY GRADED NANO-BAR BASED ON MODIFIED COUPLE STRESS THEORY

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***ABSTRACT***

*In this paper, propagation of a one-dimensional elastic stress wave in a functionally graded (FG) nano-bar is analysed based on the modified couple stress theory. It is assumed that the material properties of FG bar are distributed as an exponential function along the axial direction. The two main advantages of the modified couple stress theory over the classical couple stress theory are the inclusion of a symmetric couple stress tensor and the involvement of only one material length scale parameter. According to the modified couple stress theory, only one material length scale parameter is used to describe the size effect in nano-bar. Also, the shear stress components come from the lateral inertia effect are considered in the elastic strain energy relation. Then, the governing equations are derived using Hamilton's principle and are generally solved. Finally, effects of length scale parameter, material inhomogeneity constant and Poisson's ratio on stress wave propagation velocity and harmonic behavior of stress wave are evaluated and can be observed that using the classical continuum theory leads to considerable errors in analysis of stress wave propagation.*

***KEYWORDS****: Nano-bar, Modified Couple Stress Theory, Stress Wave Propagation, Impact Mechanic, Functionally Graded Material*

## 1.0 INTRODUCTION

Analysis of the stress wave propagation is necessary to study structures subjected to the impact loading. Therefore, the preliminary assumptions does not govern to these problems. Several basic studies are accomplished on impact mechanics problems (Fowles & Williams, 1970; Jones, 1989; Stronge, 2000; Qiao et al., 2008). However, the stress wave and generally imact problems are very important and applicable, but there are no enough studies and researchs in available about them. The one-dimensional bars are most common structure to analyse the stress wave propagation, which the stress wave propagates along the

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axial direction of them. (Anderson, 2006) obtained the longitudinal stress wave propagation of an elastic bar by using higher order rod approximations. (Shen & Yin, 2014) presented the dynamic analysis of stress waves generated by impacts on non-uniform rod structures. (Kaishin & Bin, 2001) studied the dynamic behavior of a layered orthotropic bar with rectangular cross section due to impact torque. Also, (Shariat et al., 2010)studied on other geometry for impact analysis. They analysed the stress wave in thick-walled FG cylinder with temperature-dependent material properties.

Two main approaches usually use to analyse the longitudinal wave in bars. The first of these is called to be Bernoulli-Euler rod theory (elementary wave theory). This theory assumes that deformation occurs only in the longitudinal direction and that deformed planes remain orthogonal to the deformed bar axis. The second approach is known as Love rod theory (Love, 1944). In this thoery, addition to the assumptions of the elementary wave theory, it is assumed that the plane cross sections can expand or contract in their own planes. The Love rod theory has more accuracy than Bernoulli-Euler rod theory, so, this theory is employed to describe the lateral inertia effects in the present study.

When dimension of the structures becomes very small, accuracy of classical continuum theory is decreased. Consequently, we should utilize especial theories (nonlocal theory, couple stress theory, surface effect theory) to model the small scale structures, mathematically. Modified couple stress theory proposed by (Yang et al., 2002) is one of these theories, which developed over the classical couple stress theory (Mindlin, 1964). The modified couple stress theory is a quick and simple to mathematical modelling because makes use of only one material parameter to capture the size effect. Also, this theory includes a symmetric couple stress tensor. Several studies based on modified couple stress theory in the contexts of mechanical engineering reveal the exactness and capability of this theory (Shaat et al., 2012; Ke & Wang, 2011; Salamat-talab et al., 2012; Thai & Choi, 2013).

Since small scale (micro or nano) bars can be useful and applicable in small scale devices and systems such as biosensors, atomic force microscopes (AFM), MEMS, and NEMS. But, study on stress wave propagation of nanostructures is rarely found. (Guven, 2011, 2012, 2014) presented some solutions for propagation of stress wave in small scale bars under different situations and methods.

This paper presents a modified couple stress based analysis for propagation of stress wave in longitudinally FG nano-bars using Love rod theory and Hamilton's principle. The shear stress components are considered in total strain energy relation. Finally, an explicit solution is obtained for the FG nano-bar, and effects of material length scale parameter, material inhomogeneity constant and Poisson's ratio on velocity of sress wave propagation and behavior of generated stress wave are evaluated.

## 2.0 COMPUTATIONAL METHOD

### 2.1 Functionally graded materials

Consider a solid bar with uniform cross section and area of *A* and length of *L* (see Figure 1), which material properties such as Young's modulus and density vary on the basis of an exponential function along the axial (longitudinal) direction.

$$E(x) = E_0 e^{\beta x} \tag{1}$$

$$\rho(x) = \rho_0 e^{\beta x} \tag{2}$$

where $E_0$ and $\rho_0$ are respectively Young's modulus and density of the bar at the initial point of the bar ($x$=0). Also, $\beta$ is material inhomogeneity constant.

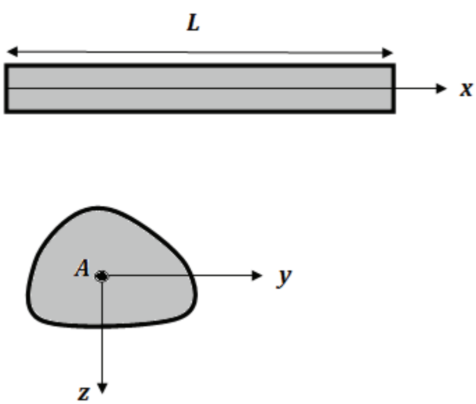


Figure 1. Shematics of geometry of coordinate system.

$$u = u(x,t), \quad \mathbf{v} = -\upsilon y \frac{\partial u}{\partial x}, \quad w = -\upsilon z \frac{\partial u}{\partial x} \tag{3}$$

where $u$, v and $w$ are respectively the $x$-, $y$- and $z$- components of the displacement on a point ($x$, $y$, $z$) on a bar cross-section. Also, $v$ is Poisson's ratio.

According to Equation (3), the non-zero components of the strain and the stress are expressed as follow:

$$\varepsilon_x = \frac{\partial u}{\partial x}, \qquad \varepsilon_y = \frac{\partial \mathbf{v}}{\partial y} = -v\frac{\partial u}{\partial x}, \qquad \varepsilon_z = \frac{\partial w}{\partial z} = -v\frac{\partial u}{\partial x},$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial \mathbf{v}}{\partial x} = -vy\frac{\partial^2 u}{\partial x^2}, \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = -vz\frac{\partial^2 u}{\partial x^2} \tag{4}$$

$$\sigma_x = E(x)\varepsilon_x = \frac{E(x)\partial u}{\partial x}, \qquad \tau_{xy} = \frac{E(x)}{2(1+v)}\gamma_{xy} = -\frac{vyE(x)}{2(1+v)}\frac{\partial^2 u}{\partial x^2},$$
$$\tau_{xz} = \frac{E(x)}{2(1+v)}\gamma_{xz} = -\frac{vzE(x)}{2(1+v)}\frac{\partial^2 u}{\partial x^2} \tag{5}$$

$\varepsilon_x$, $\varepsilon_y$ and $\varepsilon_z$ are respectively x-, y- and z- components of the normal strain. $\gamma_{xy}$ and $\gamma_{xz}$ are the shear components of the strain tensor. $\sigma_x$ is the normal stress along the x-direction. Also, $\tau_{xy}$ and $\tau_{xz}$ are the shear stresses due to the lateral inertia effect.

## 2.3 Modified couple stress theory

According to the modified couple stress theory (Yang et al., 2002), the total elastic strain energy $U$ into a region with a volume element $dV$, expresses as follow:

$$U = \frac{1}{2}\int_V (\boldsymbol{\sigma} : \boldsymbol{\varepsilon} + \boldsymbol{m} : \boldsymbol{\chi})\, dV \tag{6}$$

where $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{m}$ and $\boldsymbol{\chi}$ are Cauchy stress tensor, classical strain tensor, deviatoric part of the couple stress tensor and symmetric curvature tensor, respectively. $m$ and $\boldsymbol{\chi}$ are defined as

$$\boldsymbol{\chi} = \frac{1}{2}[(\nabla\boldsymbol{\theta})^T + (\nabla\boldsymbol{\theta})] \tag{7}$$

$$\boldsymbol{m} = \xi^2\frac{E(x)}{(1+v)}\boldsymbol{\chi} \tag{8}$$

$\boldsymbol{\theta}$ is the rotation vector and defines as

$$\boldsymbol{\theta} = \frac{1}{2}\mathbf{curl}(u) \tag{9}$$

where u is the displacement vector, which the parameters described in Equation (3) are the components of this vector. Also, is the material length scale parameter, which is mathematically the square of the ratio of the modulus of curvature to the modulus of shear and is physically regarded as material property measuring the effect of couple stress (Mindlin, 1963; Park & Gao, 2006). Since, $\xi$ is a function of the material, so, must be varied along the axial direction similar to Young's modulus and density. But for simplicity case and parametric study, similar to several studies accomplished on FG nanostructures (Reedy, 2011; Jung et al., 2014), this parameter assumes constant.

Substituting Equation (3) into Equation (9), the non-zero components of rotation vector are obtained as:

$$\boldsymbol{\theta}_y = \frac{1}{2}vz\frac{\partial^2 u}{\partial x^2}, \qquad \boldsymbol{\theta}_z = -\frac{1}{2}vy\frac{\partial^2 u}{\partial x^2} \tag{10}$$

and by substituting Equation (10) into Equation (7), we have:

$$\chi_{xy} = \frac{1}{4}vz\frac{\partial^3 u}{\partial x^3} = \chi_{yx}, \qquad \chi_{xz} = -\frac{1}{4}vy\frac{\partial^3 u}{\partial x^3} = \chi_{zx} \tag{11}$$

also from Eqs. (8) and (11), the non-zero components of tensor *m* obtain as:

$$\boldsymbol{m}_{xy} = \frac{1}{4}\xi^2\frac{vzE(x)}{(1+v)}\frac{\partial^3 u}{\partial x^3}, \qquad \boldsymbol{m}_{xz} = -\frac{1}{4}\xi^2\frac{vyE(x)}{(1+v)}\frac{\partial^3 u}{\partial x^3} \tag{12}$$

## 2.4 Equation of motion

In this study, the equation of motion is derived using Hamilton's principle. First, the virtual strain energy and the virtual kinetic energy are obtained as:

$$\delta U = \int_V (\boldsymbol{\sigma}:\boldsymbol{\delta\varepsilon} + \boldsymbol{m}:\boldsymbol{\delta\chi})dV = \int_V (\sigma_x\delta\varepsilon_x + \tau_{xy}\delta\gamma_{xy} + \tau_{xz}\delta\gamma_{xz} + 2m_{xy}\delta\chi_{xy} + 2m_{xz}\delta\chi_{xz})dV \tag{13}$$

$$\delta T = \frac{1}{2}\rho(x)\int_V (\dot{u}^2 + \dot{\mathbf{v}}^2 + \dot{w}^2)dV \tag{14}$$

Now, by using Hamilton's principle as $\int_{t_1}^{t_2}\delta(T-U)$, where is variation symbol. Finally, the equation of motion is derived as follow:

$$\frac{\partial E(x)}{\partial x}\frac{\partial u}{\partial x}+E(x)A\frac{\partial^2 u}{\partial x^2}-\frac{v^2 I}{2(1+v)}\frac{\partial^2 E(x)}{\partial x^2}\frac{\partial^2 u}{\partial x^2}-\frac{v^2 I}{(1+v)}\frac{\partial E(x)}{\partial x}\frac{\partial^3 u}{\partial x^3}-\frac{v^2 E(x) I}{2(1+v)}\frac{\partial^4 u}{\partial x^4}$$
$$-\rho(x)A\frac{\partial^2 u}{\partial t^2}+v^2 I\frac{\partial \rho(x)}{\partial x}\frac{\partial^3 u}{\partial x\partial t^2}+v^2\rho(x)I\frac{\partial^4 u}{\partial x^2\partial t^2}=0 \tag{15}$$

In this analysis, a harmonic longitudinal wave propagating along the axial direction is considered, which can be expressed in the complex form as:

$$u=\tilde{u}e^{ik(x-ct)} \tag{16}$$

where k, c and $\tilde{u}$ are the wave number, the mean velocity of wave propagation in an FG nano-bar and the wave amplitude, respectively. Similar to what was mentioned for the material length scale parameter $\xi$ , the velocity of wave propagation in an FG nano-bar must be varied as a function of *x*-component (Preferably exponentially), but this velocity is assumed to be constant and term of the mean velocity in the bar is used for it.

Substituting Eqs. (1), (2) and (16) into Equation (15), the equation of motion achieves as:

$$-E_0A+\frac{v^2\beta^2E_0I}{2(1+v)}-\frac{v^2E_0I}{2(1+v)}k^2+\frac{\xi^2v^2\beta^2E_0I}{4(1+v)}k^2-\frac{\xi^2v^2E_0I}{8(1+v)}k^4+\rho_0Ac^2+v^2\rho_0Ic^2k^2=0 \tag{17}$$

By a direct solution, we have:

$$c=\sqrt{\frac{E_0}{\rho_0}\left(\frac{1+\dfrac{4v^2r^2k^2-4v^2r^2\beta^2+\xi^2v^2r^2k^4-2\xi^2v^2r^2\beta^2k^2}{8(1+v)}}{1+v^2r^2k^2}\right)} \tag{18}$$

where $r=\sqrt{I/A}$ states the gyration radius and *I* is the polar moment of inertia with respect to the z-axis. Thus, for the circular cross section, we have $r=\frac{a}{\sqrt{2}}$. Equation (18) presents the mean velocity of longitudinal stress wave propagation for an FG nano-bar by consideration of Poisson's effect. Now, this general relation can be derived for some particular cases. For example, when the nano-bar made of a homogeneous material with constant Young's modulus $E_0$ and constant density $\rho_0$ ($\beta=0$ ), we have:

$$c = \sqrt{\frac{E_0}{\rho_0}\left(\frac{1 + \frac{4v^2r^2k^2 + \xi^2v^2r^2k^4}{8(1+v)}}{1 + v^2r^2k^2}\right)} \tag{19}$$

To obtain the mean velocity of stress wave propagation based on classical theory, It is enough that the material length scale parameter comes from the modified couple stress theory sets to zero ($\xi = 0$). So, we have:

$$c = \sqrt{\frac{E_0}{\rho_0}\left(\frac{1 + \frac{4v^2r^2k^2 - 4v^2r^2\beta^2}{8(1+v)}}{1 + v^2r^2k^2}\right)} \tag{20}$$

By disregard the Poisson's effect (v=0), Equation (18) rewrites as follow:

$$c = c_0 = \sqrt{\frac{E_0}{\rho_0}} \tag{21}$$

where $c_0$ is the velocity of stress wave propagation in a simple Bernoulli-Euler bar.

## 3.0 RESULTS AND DISCUSSIONS

In this paper, a general solution for different cross sections is done. This section presents numerical results of the stress wave propagation in an FG nano-bar made of circular cross section with radius a=0.34 nm. Effects of size, heterogeneity of material and Poisson's ratio on the velocity and behavior of the stress wave are evaluated.

Figure 2 shows the non-dimensional mean velocity of stress wave propagation versus the wave number with different material length scale parameters, where $k^* = ka$ is non-dimensional wave number. In this figure, the size effect is clearly shown and it is observed that by increasing the material parameter at a given radius, the mean velocity of stress wave propagation is increased. This exposes the size-dependent behavior of nano-bars subjected to excitation of the harmonic stress wave. $\xi = 0$ in this figure expresses the non-dimensional mean velocity of stress wave propagation based on the classical theory. As can be seen, the classical theory has considerable errors to estimate the velocity of stress wave propagation and this theory can be useful for macro scale structures.

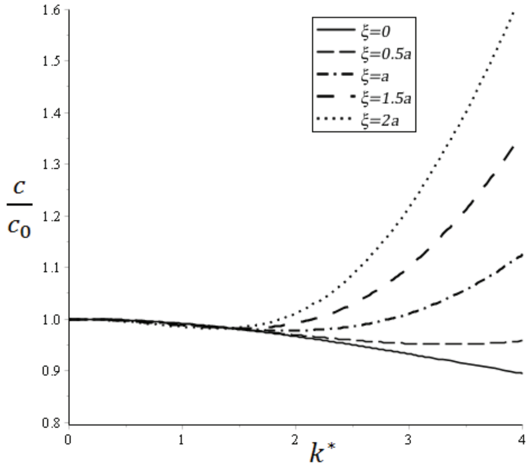


Figure 2. Effect of material parameter on velocity of stress wave propagation with $\beta = 1$ and v=0.25.

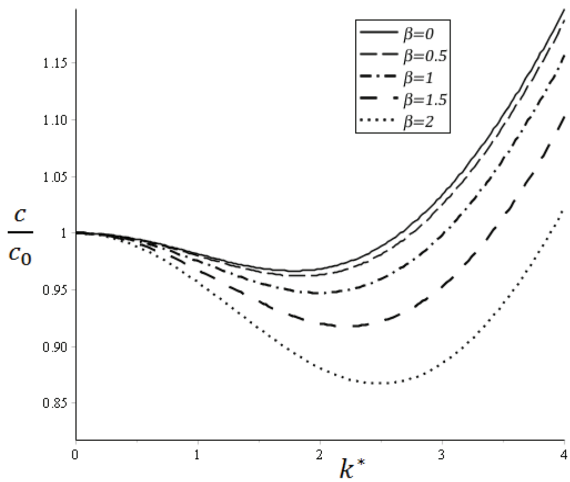


Figure 3. Effect of the material inhomogeneity constant on velocity of stress wave propagation $\xi = a$ and v=0.25.

Figure 3 illustrates the effect of the material inhomogeneity constant $\beta$ on velocity of stress wave propagation. This figure shows that increasing the material inhomogeneity constant leads to decreasing the mean velocity of stress wave propagation. In fact, the velocity of stress wave propagation is averagely reduced when the heterogeneity of material increases.

Poisson's effect on velocity of stress wave propagation expresses in Figure 4. For small non-dimensional wave number (approximately less than 3), the velocity of stress wave propagation is decreased by increasing Poisson's ratio, while for larger non-dimensional wave numbers, the velocity of stress wave propagation is increased by increasing Poisson's ratio. Also, when the lateral effect is neglected (v=0), the velocity of stress wave propagation becomes equal to a constant value (velocity of stress wave propagation in a homogeneous Bernoulli-Euler bar). As can be seen in Figs. 2-4, for large non-dimensional wave numbers $k^*$, the velocity of stress wave propagation is increased by increasing $k^*$, and increasing of $k^*$ for small non-dimensional wave numbers leads to decreasing the velocity of stress wave propagation.

According to Equation (5), the stress wave made in the nano-bar obtains as $\sigma_x = \sigma_0 e^{\beta x} e^{ik(x-ct)}$, where $\sigma_0 = E_0 \tilde{u} k$. Variations of real part of the non-dimensional stress wave against non-dimensional wave number with different material length scale parameters under $\beta = 1$, $v$=0.25, $x$=10$a$ and $t$=0.1$s$ are shown in Figure 5. In this figure, the stress wave behavior is completely harmonic except for very small values of $k^*$. This is because of the fact that when the wave number tends to zero then the incoming wave loses its harmonic vitality and becomes a constant wave (Equation (16)). Moreover, by increasing $k^*$, the wave length of stress wave is decreased because of the wave number introduced in Equation (16) relates with inverse of the incoming wave length. Also, the size effect on stress wave is studied and it is observed that by increasing the material parameter $\xi$, the stress wave propagated in nano-bar starts its harmonic behavior earlier and leads to increasing of stress wave intensity. Similar to what was mentioned for Figure 5, the material inhomogeneity constant and Poisson's ratio have similar effect on harmonic behavior of the stress wave (Figures 6 and 7).

Maximum shear stress wave made in nano-bar with circular cross section is as $\tau_m = \tau_0 v k^* e^{\beta x} e^{ik(x-ct)}$ (Equation 5), where $\tau_0 = G_0 \tilde{u} k$ ($G = E/2(1+v)$). It should be noted that for circular cross section, we have: $\tau_{xy} = \tau_{xz}$. The harmonic behavior of non-dimensional shear stress wave against non-dimensional wave number is shown in Figure 8. By increasing $k^*$, intensity and amplitude of the shear stress increases. This is because of the fact that the shear stress made in nano-bar is caused by lateral inertia, therefore, this is dependent on radius of bar. Consequently, by increasing $k^*$ at a given wave number, the radius of bar increases. So, by increasing $k^*$, amplitude of the shear stress wave increases. Because the behavior of the shear stress wave versus the material parameter, material inhomogeneity constant and Poisson's ratio is similar to axial stress wave, evaluation of theses behaviors are not considered.

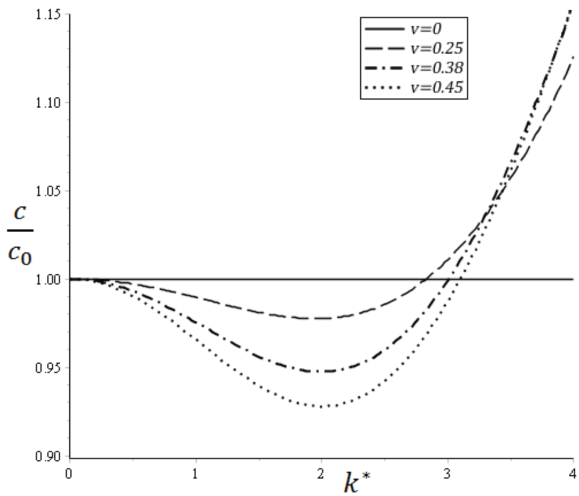


Figure 4. Poisson's effect on velocity of stress wave propagation with $\beta = 1$ and $\xi = a$.

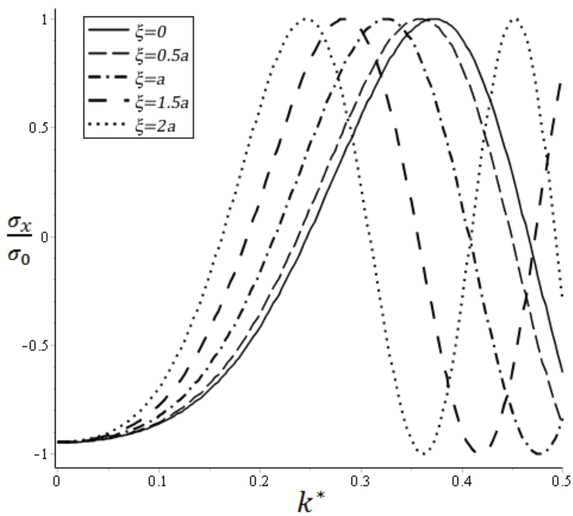


Figure 5. Behavior of non-dimensional axial stress wave versus non-dimensional wave number with different material length scale parameter under $\beta = 1$ , $v$=0.25, $x$=10$a$ and $t$=0.1$s$.

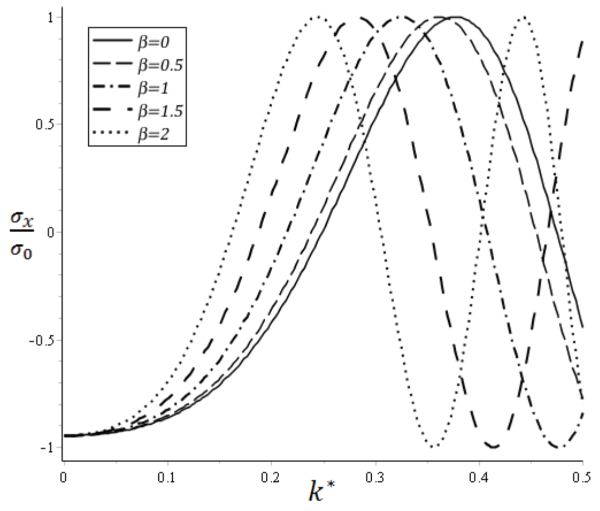


Figure 6. Behavior of non-dimensional axial stress wave versus non-dimensional wave number with different material inhomogeneity constant under $\xi = a$, $v$=0.25, $x$=10$a$ and $t$=0.1$s$.

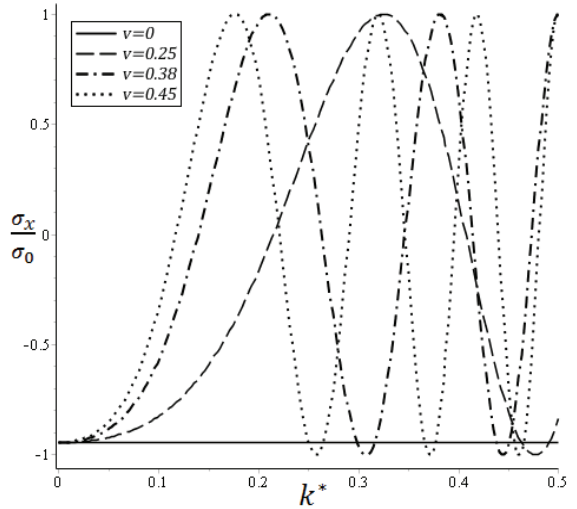


Figure 7. Behavior of non-dimensional axial stress wave versus non-dimensional wave number with different Poisson's ratio under $\beta = 1$, $\xi = a$, $x$=10$a$ and $t$=0.1$s$

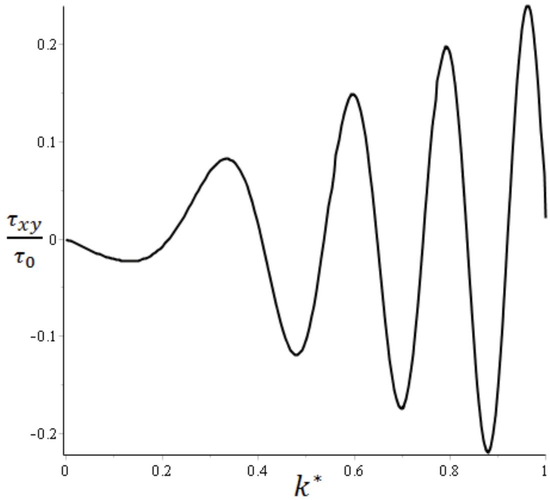


Figure 8. Behavior of non-dimensional shear stress wave versus non-dimensional wave number with different material length scale parameter under $\beta = 1$ , $\xi = a$ , $v$=0.25, $x$=10$a$ and $t$=0.1$s$.

## 4.0 CONCLUSIONS

Propagation of longitudinal stress wave in an FG nano-bar which graded longitudinally is studied in this paper. The equation of motion is derived using modified couple stress theory, Hamilton's principle and Love rod theory. The velocity of stress wave propagation of the nano-bar is obtained as a function of Poisson's ratio, material length scale parameter and material inhomogeneity constant by a direct solution of the equation of motion. The following results are concluded from analysis of the stress wave by the mentioned parameters. Behavior of the stress wave propagation of the nano-bar is a size-dependent behavior and this dependency exposes using the material length scale parameter $\xi$. The numerical results show that by increasing the $\xi$, the velocity and intensity of the stress wave are increased. Moreover, neglecting of material length scale parameter (use of classical theory, $\xi = 0$) leads to considerable errors. Thereupon, the inability of the classical theory to analyse the micro/nanostructures is confirmed. The non-dimensional stress wave against the non-dimensional wave number behaves harmoniously and by increasing non-dimensional wave number $k^*$ the wave length of the stress wave is decreased. Also, when $k^*$ tend to zero, the stress wave loses its harmonic behavior and consequently the stress wave becomes constant.

By variation of the material inhomogeneity constant $\beta$ in graded structures can be derived the velocity of the wave and behavior of the stress wave. The results show that the graded materials have a less velocity than homogeneous materials($\beta = 0$ ). Also, by increasing , velocity of the stress wave propagation is decreased, but the harmonic behavior of the stress wave occurs earlier.

The results show that neglecting the lateral effct (v=0) leads to make the considerable error in the impact behavior of structures. For small $k^*$, increasing the Poisson's ratio estimates less velocity for the stress wave; and for large $k^*$, increasing the Piosson's ratio leads to increase the velocity of stress wave propagation. Also, increasing the Poisson's ratio leads to the generated stress wave arrives to its harmonic behavior earlier.

## 5.0 REFERENCES

Anderson, S. P. (2006). Higher-order rod approximations for the propagation of longitudinal stress waves in elastic bars. *Journal of Sound and Vibration, 290*, 290-308.

Fowles, R., & Williams, R. F. (1970). Plane stress wave propagation in solids. *Journal of Applied Physics, 41*, 360-363.

Güven, U. (2011). The investigation of the nonlocal longitudinal stress waves with modified couple stress theory. *Acta Mechanica, 221*, 321-325.

Güven, U. (2012). A more general investigation for the longitudinal stress waves in microrods with initial stress. *Acta Mechanica, 223*, 2065–2074.

Güven, U. (2014). A generalized nonlocal elasticity solution for the propagation of longitudinal stress waves in bars. *European Journal of Mechanics A/ Solids, 45*,75-79.

Jones, N. (1989). *Stractural Impact* (1st ed.). London, UK: Cambridge University.

Jung, W. Y., Han, S. Ch., & Park, W. T. (2014). A modified couple stress theory for buckling analysis of S-FGM nanoplates embedded in Pasternak elastic medium. *Composites Part-B Engineering, 60,* 746-756.

Kaishin, L., & Bin, L. (2001). A numerical solution of torsional stress wave propagation in layered orthotropic bar of rectangular cross-section. *International Journal of Solids and Structures, 38*, 8929-8940.

Ke, L. L., & Wang, Y. Sh. (2011). Size effect on dynamic stability of functionally graded microbeams based on a modified couple stress theory. *Composites Structures, 93*, 342-350.

Love, A. E. H. (1944). *A Treatise on the Mathematical Theory of Elasticity*. New York, USA: Dover Publications.

Mindlin, R. D. (1963). Influence of couple-stresses on stress concentrations. *Experimental Mechanics, 3*, 1–7.

Mindlin, R. D. (1964). Microstructure in linear elasticity. *Archive for Rational Mechanics and Analysis, 16*, 51–78.

Park, S. K., & Gao, X. L. (2006). Bernoulli–Euler beam model based on a modified couple stress theory. *Journal of Micromechanics and Microengineering, 16*, 2355–2359.

Qiao, P., Yang, M., & Bobaru, F. (2008). Impact mechanics and high-energy absorbing materials: Review. *Journal of Aerospace Engineering, 21*(4), 235-248.

Reddy, J. N. (2011). Microstructure-dependant couple stress theories of functionally graded beams. *Journal of the Mechanics and Physics of Solids, 59*, 2382–2399.

Salamat-talab, M., Nateghi, A., & Torabi, J. (2012). Static and dynamic analysis of third-order shear deformation FG micro beam based on modified couple stress theory. *International Journal of Mechanical Sciences, 57*, 63-73.

Shaat, M., Mahmoud, F. F., Alshorbagy, A. E., Alieldin, S. S., & Meletis, E. I. (2012). Size-dependent analysis of functionally graded ultra-thin films. *Structural Engineering Mechanics, 44*(4), 431-448.

Shariat, M., Khaghani, M., & Lavasani, S. M. H. (2010). Nonlinear thermoelasticity, vibration, and stress wave propagation analyses of thick FGM cylinders with temperature-dependant material properties. *European Journal of Mechanics A/Solids, 29*, 378-391.

Shen, Y., & Yin, X. (2014). Dynamic substructure analysis of stress waves generated by impacts on non-uniform rod structures. *Mechanism and Machine Theory, 74*, 154-172.

Stronge, W. J. (2000). *Impact Mechanics* (1st ed.). London, UK: Cambridge University.

Thai, H. T., & Choi, D. H. (2013). Size-dependent functionally graded Kirchhoff and Mindlin plate models based on a modified couple stress theory. *Composites Structures, 95*, 142–153.

Yang, F., Chong, A. M., Lam, D. C. C., & Tong, P. (2002). Couple stress based strain gradient theory of elasticity. *International Journal of Solids and Structures, 39*, 2731-2743.