A CLOSED-FORM SOLUTION FOR THE SIMILARITY TRANSFORMATION PARAMETERS OF TWO PLANAR POINT SETS

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ABSTRACT

Image registration covers the set of techniques used in matching images of the same scene. A subset of the image registration problem, identifying the parameters in a similarity transformation, has emerged as useful in a recently defined area of machine design: designing mechanisms for rigid-body shape-change. First, this brief paper shows a potential use for image registration techniques outside the field of machine vision. Second, it presents a closed-form solution for the similarity transformation parameters when the point sets to be matched are restricted to two-dimensional space as is needed in the aforementioned design problem.

KEYWORDS*: Image registration, Similarity transformation, Closed-form solution, Planar*

1.0 INTRODUCTION

As identified in Zitová and Flusser (2003), a critical step in image registration is "transform model estimation." In this step, one attempts to identify the mapping functions that will *best* align points on a sensed image with the corrseponding points on a reference image. In some cases, this mapping function is restricted to be a similarity transformation,

$$V_i = cAv_i + d, \tag{1}$$

where v_i are the points in the sensed image, c is a scaling factor, A is a rotation matrix, and d is a displacement vector. The desired values of c, A, and d are those that minimize

$$D(A, d, c) = \sum^{n} |V_i - U_i|^2, \tag{2}$$

where U_i=(U_{xi},U_{yi}) are the points in the reference image and Vi=(Vxi,Vyi) are the points in the transformed image.

In the design of rigid-body shape-changing mechanisms, a designer is presented with two (or more) target profiles. A target profile is a piecewise linear representation of a desired shape for a feature in a mechanism. To complete the design of the mechanism, we seek an "average" representation of these profiles. Three such profiles are shown in Figure 1a. Denoting the points on the "reference profile" as U_i and those on one of the other two profiles (the dash-dot profile, for example) as v_i, we seek the values of A and d that solve Eqs. (1) and (2) under the restriction $c = 1$. This restriction is necessary due to the consideration of these points as representing rigid bodies. Figure 1b shows both of the target profiles aligned with the reference profile in this way. Continuing with the design of a rigid-body shape-changing mechanism, a "mean profile" is generated from the geometric center of each U_i and the shifted v_i, as shown in Figure *1c*. Using additional similarity transformations, the mean profile is now shifted to the locations that solve Eqs. (1) and (2) (again with $c = 1$) relative to each of the target profiles, as shown in Figure 1d. Thus, similarity transformations are needed both to align all of the target profiles in order to create a mean segment and to move this mean segment back to the locations nearest the original target profiles.

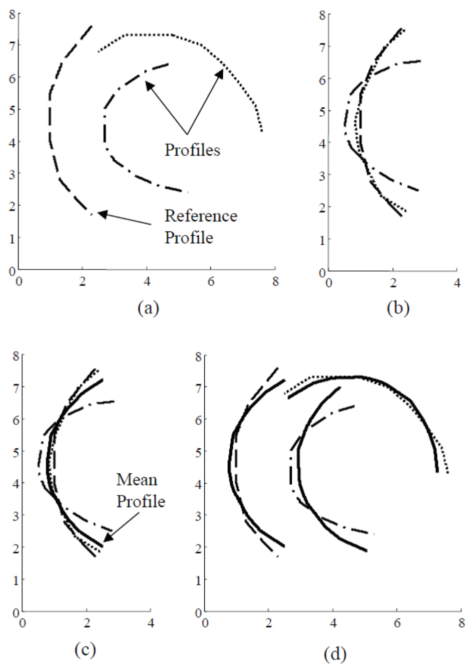


FIGURE 1
(a) Three target profiles, with one deemed the reference profile. (b) Two profiles are transformed to the reference by a similarity transformation (with $c = 1$). (c) The mean profile. (d) The mean profile transformed back to the original profile locations

FIGURE 2
(a) Rigid bodies connected with revolute joints form a chain to closely approximate the profiles. (b) A mechanism design to move the chain of rigid bodies between the three

If the design goal were to find a single body to approximate the three target profiles, we would proceed to determine the dimensions of a mechanism that could guide the body (the mean segment) into the three positions shown in Figure 1d. Should this solution not reproduce the orginal profiles with adequate accuracy, though, the process described above can be performed on individual portions of the target profiles. Figure 2a shows an example in which the three target profiles are considered to be composed of four segments. Each segment now does a better job of approximating a shorter section of the original target profile, and combining these segments into a single chain connected by revolute joints produces a more accurate matching of the original target profiles. Figure 2b shows the complete mechanism that can be used to move the chain between the three profiles. Note that increasing the number of segments increases the complexity (the number of links) of the mechanism that guides the segments between the three profiles. The details of this process are found in Murray et.al. (2008) and Persinger et.al. (2009).

The solution to Equations. (1) and (2) has been thoroughly addressed in the literature. Notable among this work is that of Horn (1987), Horn et.al. (1988) and Arun et.al. (1987) who address the problem for three-dimensional point sets. Umeyama (1991) noted and corrected the problem of an A resulting in a reflection (instead of a rotation) for what he refers to as “corrupted data.” Both Umeyama's approach and that of Wen et.al. (2006) work on sets containing points with an arbitrary number of dimensions. Central to the solutions in these papers is the use of the singular value decomposition (SVD). Rapid and robust numerical techniques for determining the SVD are well established in the literature, for example, (Forsythe, 1977) and (Golub, 1983). Even more recently, spanning graphs have been proposed due to the efficiency of the associated numerical method (Sabuncu, 2008).

When Ui and vi are points confined to a plane, though, Equations. (1) and (2) are shown in this technical brief to yield a closed-form solution. The details of this solution are presented in the next section. Also of note, the correct values of A, d and c are generated even in the presence of “corrupted data.” Finally, the example from Umeyama (1991) is used to verify this methodology.

2.0 THE CLOSED-FORM SIMILARITY TRANSFORMATION SOLUTION

We seek the values of c, θ in the rotation matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{3}$$

and the translation vector d given by

$$d = \begin{Bmatrix} d_x \\ d_y \end{Bmatrix} \tag{4}$$

in Equation (1) such that

$$\begin{aligned} D &= \frac{1}{n}\sum_{i=1}^{n} |V_i - U_i|^2 \\ &= \frac{1}{n}\sum_{i=1}^{n} \left((V_{xi} - U_{xi})^2 + (V_{yi} - U_{yi})^2\right) \end{aligned} \tag{5}$$

is minimized. As all summations in the paper are from i=1 to n, the summation bounds are omitted from all further equations.

The c, θ and d values may be obtained through solving the simultaneous set of equations

$$\frac{\partial D}{\partial d} = \vec{0}, \frac{\partial D}{\partial \theta} = 0, \frac{\partial D}{\partial c} = 0. \tag{6}$$

Substituing Equation (1) into Equation 5 and expanding gives

$$\begin{aligned} D &= \frac{1}{n}\sum |cAv_i + d - U_i|^2 \\ &= \frac{1}{n}\sum (c^2 v_i^T v_i + d^T d + U_i^T U_i + 2cd^T A v_i - 2cU_i^T A v_i - 2d^T U_i). \end{aligned} \tag{7}$$

The first of the simultaneous equations in (6) yields

$$\frac{\partial D}{\partial d} = \sum (d + cAv_i - U_i) = \vec{0}. \tag{8}$$

Noting the definitions

$$\sum v_i = v_t = \{v_{xt} \quad v_{yt}\}^T \tag{9}$$

and

$$\sum U_i = U_t = \{U_{xt} \quad U_{yt}\}^T, \tag{10}$$

Eq. 8 may be solved for d as

$$d = \frac{1}{n}(U_t - cAv_t). \tag{11}$$

The derivative of A is

$$\frac{\partial D}{\partial \theta} = \begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix} = A\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = A\hat{I}, \tag{12}$$

and the second of the simultaneous equations in (6) yields

$$\frac{\partial D}{\partial \theta} = \sum\left(d^T A\hat{I}v_i - U_i^T A\hat{I}v_i\right) = 0 = \sum(d - U_i)^T A\hat{I}v_i. \tag{13}$$

Substituting Equation (11) into Equation (13) eliminates d, and recognizing that as a rotation matrix, $A^T = A^{-1}$,

$$\sum\left(\frac{1}{n}U_t - \frac{1}{n}cAv_t - U_i\right)^T A\hat{I}v_i = \sum\left(\frac{1}{n}U_t^T A\hat{I}v_i - \frac{1}{n}cv_t^T\hat{I}v_i - U_i^T A\hat{I}v_i\right) = 0. \tag{14}$$

Observe that

$$\sum\frac{1}{n}cv_t^T\hat{I}v_i = \frac{1}{n}cv_t^T\hat{I}v_t = 0 \tag{15}$$

due to the skew-symmetry of $\hat{I}$. Equation (14) becomes

$$\sum\left(\frac{1}{n}U_t^T A\hat{I}v_i - U_i^T A\hat{I}v_i\right) = \frac{1}{n}U_i^T A\hat{I}v_t - \sum U_i^T A\hat{I}v_i = 0. \tag{16}$$

We can expand Equation (16) as

$$\frac{1}{n}\{U_{xt} \quad U_{yt}\}\begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}\begin{Bmatrix} v_{xt} \\ v_{yt} \end{Bmatrix} = \sum\{U_{xi} \quad U_{yi}\}\begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}\begin{Bmatrix} v_{xi} \\ v_{yi} \end{Bmatrix}. \tag{17}$$

By expanding Equation (17) and solving for θ,

$$\theta = ATAN2(C_1, C_2), \tag{18}$$

where

$$\begin{aligned} C_1 &= \frac{1}{n}\left(U_{xt}v_{yt} - v_{xt}U_{yt}\right) - \sum\left(U_{xi}v_{yi} - v_{xi}U_{yi}\right), \\ C_2 &= \sum\left(U_{xi}v_{xi} + v_{yi}U_{yi}\right) - \frac{1}{n}\left(U_{xt}v_{xt} + v_{yt}U_{yt}\right), \end{aligned} \tag{19}$$

and ATAN2 is a four-quadrant inverse tangent function. This determines the proper value of θ, so via Equation (3), A is now known. Since A is defined as a rotation matrix, it cannot result in a reflection in the presence of "corrupted data," a possibility when an SVD is used as noted and corrected for by Umeyama (Umeyama, 1991).

From the third of the simultaneous equations in (6),

$$\frac{\partial D}{\partial c} = \frac{1}{n}\sum\left(2cv_i^T v_i + 2d^T Av_i - 2U_i^T Av_i\right) = 0 \tag{20}$$

from which d may be eliminated by substituting Equation (11),

$$\sum\left(cv_i^T v_i + \frac{1}{n}(U_t - cAv_t)^T Av_i - U_i^T Av_i\right) = 0. \tag{21}$$

Expanding,

$$c\left(\sum v_i^T v_i - \frac{1}{n}v_t^T v_t\right) = \sum U_i^T Av_i - \frac{1}{n}U_t^T Av_t. \tag{22}$$

Recalling that A is known, c may be found as

$$c = \frac{E_1 - E_2}{E_3 - E_4}, \tag{23}$$

where

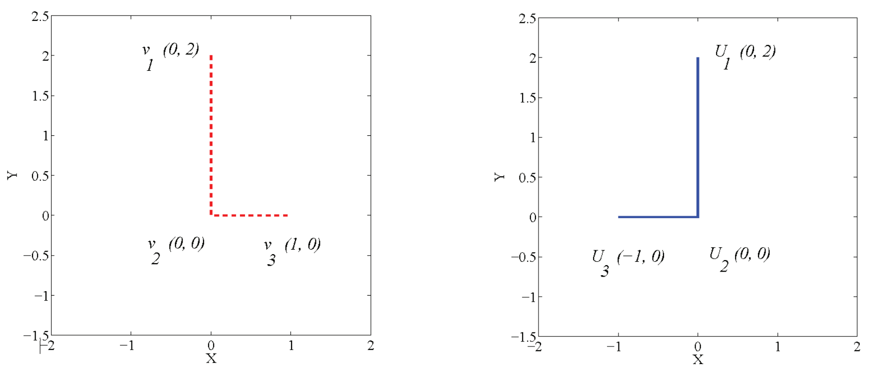
$$
\begin{aligned}
E_1 &= \sum \Big(U_{xi}\big(v_{xi}\cos\theta - v_{yi}\sin\theta\big) + U_{yi}\big(v_{xi}\sin\theta - v_{yi}\cos\theta\big)\Big),\\
E_2 &= \frac{1}{n}\Big(U_{xt}\big(v_{xt}\cos\theta - v_{yt}\sin\theta\big) + U_{yt}\big(v_{xt}\sin\theta - v_{yt}\cos\theta\big)\Big),\\
E_3 &= \sum \big(v_{xi}^2 + v_{yi}^2\big),\\
E_4 &= \frac{1}{n}\big(v_{xt}^2 + v_{yt}^2\big).
\end{aligned} \tag{24}
$$

Substituting c and A into Equation (11) produces the value of d, and the similarity transformation is now known.

3.0 EXAMPLE

Figures 3a and 3b show the example presented in Umeyama (Umeyama, 1991) with data that is considered corrupted as the methods established prior to Umeyama yielded a reflection rather than a rotation. The reference points U_i are (0,2), (0,0), and (-1,0). The sensed points v_i to which the similarity transformation is to be applied are (0,2), (0,0), and (1,0). Note that the order of the points is important since the n^{th} point of the first list corresponds to the n^{th} point of the second. Using Equations (9) and (10),

$$
v_t = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}, U_t = \begin{Bmatrix} -1 \\ 2 \end{Bmatrix}. \tag{25}
$$



(a) (b)

FIGURE 3
(a) The points vi to undergo the similarity transformation. (b) The reference points U_i.

Using Equation (18), the angle θ=-33.6901°, and the corresponding rotation matrix is

$$
A = \begin{bmatrix} 0.8321 & 0.5547 \\ -0.5547 & 0.8321 \end{bmatrix} \tag{26}
$$

From Equation (23), the scaling factor is c = 0.7211. Finally, Equation (11) is used to find

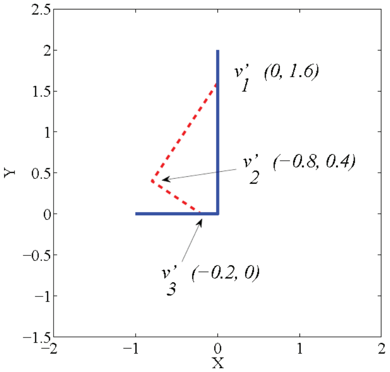
$$d = \begin{Bmatrix} -0.8000 \\ 0.4000 \end{Bmatrix} \tag{27}$$

These values match those reported by Umeyama (1991). The result of using this similarity transformation to align v_i with Ui is shown in Figure 4.

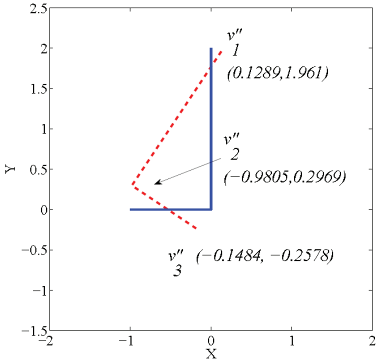
Note that for the rigid-body shape-changing calculation, Eq. (23) is not used and c = 1. In this case, Equation (11) yields

$$d = \begin{Bmatrix} -0.9805 \\ 0.2969 \end{Bmatrix}. \tag{28}$$

The result of using this similarity transformation to align the v_i with the U_i is shown in Figure 4.



(a)



(b)

FIGURE 4

(a) The scaled, rotated and translated v_i that align with the U_i for the optimal c, A, and d. (b) The scaled, rotated and translated v_i that align with the U_i for the optimal c=1.

4.0 CONCLUSION

When designing a rigid-body shape-changing mechanism, a problem familiar to the image registration community arises - obtaining the parameters in a similarity transformation that minimizes the sum of the squares of the distances between points on a reference image and the corresponding points in a sensed image. This problem has been thoroughly addressed for point sets in an arbitrary number of dimensions and for corrupted data.

In the design of planar shape-changing mechanisms, the point sets are confined to two dimensions. Restricting the problem in this way, a closed-form solution to the similarity transformation problem was developed. Even though the scaling parameter can be determined as part of this derivation, the rigid-body criterion requires that we consider the problem as having no scaling factor ($c = 1$). Additionally, this design problem uses these similarity transformations a large number of times, in both aligning data sets to create an average version of these sets, and then in aligning this average set with the original data. As the original data set may be parsed in multiple ways, typical problems require thousands of such transformations.

5.0 ACKNOWLEDGMENT

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